

# Approximating rational Bézier curves by constrained Bézier curves of arbitrary degree

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## Abstract

In this paper, we propose a method to obtain a constrained approximation of a rational Bézier curve by a polynomial Bézier curve. This problem is reformulated as an approximation problem between two polynomial Bézier curves and is solved by linear least-squares methods. The efficiency of the proposed method is tested using some examples.

*Keywords:* Rational Bézier curves, Approximation, Bézier curves, Least-squares method

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## 1. Introduction

Rational Bézier curves play a significant role in Computer Aided Design systems. However, because the forms of derivatives are quite complex and integrals do not exist for high-degree rational Bézier curves, the problem of approximating rational functions with polynomials has been raised and studied. Sederberg and Kakimoto [1] presented the hybrid polynomial approximation to rational curves for the first time in 1991. Wang et al.[2]

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presented Hermite polynomial approximations to rational Bézier curves and investigated the convergence condition for the polynomial approximation of rational functions and rational curves. Floater [4] constructed a high-order approximation of rational curves using polynomial curves. Huang [3] presented a simple method for approximating a rational Bézier curve with a Bézier curve sequence based on degree elevation. Lee [5] converted a polynomial approximation of a rational Bézier curve to an approximation about two polynomial curves and obtained an approximate solution by solving a linear least-squares problem. Recently, the sample-based polynomial approximation of rational Bézier curves and approximating conic sections by constrained Bézier curves of arbitrary degree were investigated by Lu [6] and Hu [7], respectively.

The above-mentioned methods have their advantages and disadvantages. The conversion of a polynomial approximation of a rational Bézier curve to an approximation between two polynomial Bzier curves and solving the problem based on a least-squares method has the following advantages: First, the degree of approximation to the polynomial curve is arbitrary; second, for polynomial curves, the integral expression based on a least-squares method is analytic; third, because there exist a variety of methods for solving linear equations, computing the control points of a Bézier curve is quick and stable. In this paper, we extend the method presented in paper [5] to the constrained approximation of a rational Bézier curve. Our method also differs from that presented in[5] in that we don't use a degree elevation formula for Bézier curves.

The paper is structured as follows. Section 2 presents some basic concepts

and properties regarding the problem of the constrained Bézier approximation of a rational Bézier curve. Section 3 brings an complete solution to the problem formulated above in the  $L_2$  norm. Section 4 presents some numerical examples to verify the accuracy and effectiveness of the method.

## 2. Preliminaries

### 2.1. Definitions and Properties

A standard-form  $n$ th degree rational Bézier curve is defined as follows:

$$\mathbf{P}(t) = \frac{\mathbf{x}(t)}{\omega(t)} = \frac{\sum_{i=0}^n \omega_i \mathbf{p}_i B_i^n(t)}{\sum_{i=0}^n \omega_i B_i^n(t)}, \quad t \in [0, 1], \quad (1)$$

where  $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$  are the Bernstein polynomials,  $\mathbf{p}_i \in \mathbb{R}^d$  are control points and  $\omega_i \in \mathbb{R}^+$  ( $\omega_0 = \omega_n = 1$ ) are the scalar weights.

The following are pertinent properties used in this paper:

**Property 1.** *The product of two Bernstein polynomials and its integral satisfies [8]*

$$B_i^n(t) B_j^m(t) = \frac{\binom{n}{i} \binom{m}{j}}{\binom{m+n}{i+j}} B_{i+j}^{m+n}(t), \quad (2)$$

$$\int_0^1 B_i^n(t) B_j^m(t) dt = \frac{\binom{n}{i} \binom{m}{j}}{(m+n+1) \binom{m+n}{i+j}}. \quad (3)$$

**Property 2.** *Let the two polynomials  $f(t)$  and  $g(t)$  of degree  $m$  and  $n$  with coefficients  $f_i^m$  and  $g_i^n$  be as follows [9]*

$$f(t) = \sum_{i=0}^m f_i^m B_i^m(t), \quad g(t) = \sum_{i=0}^n g_i^n B_i^n(t).$$

Their product is a degree  $m + n$  polynomial

$$f(t)g(t) = \sum_{i=0}^{m+n} \left( \sum_{j=\max(0, i-n)}^{\min(m, i)} \frac{\binom{n}{j} \binom{m}{i-j}}{\binom{m+n}{i} f_j^m g_{i-j}^n} \right) B_i^{m+n}(t). \quad (4)$$

**Property 3.** For each appropriate function  $f(x)$ , there is a unique least-squares polynomial approximation of degree at most  $n$  that minimizes the function[10].

## 2.2. Statement of the approximation problem

The problem of approximating rational Bézier curves in (1) by constrained Bézier curves of arbitrary degree is that of finding control points  $\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_m$ , which define a Bézier curve of degree  $m$

$$\mathbf{Q}(t) = \sum_{i=0}^m B_i^m(t) \mathbf{q}_i, \quad (5)$$

such that the following two conditions are satisfied simultaneously[7]:

- 1) Bézier curve  $\mathbf{Q}(t)$  has  $C^0, C^1$  or  $C^2$  continuity at both endpoints of the rational Bézier curve  $\mathbf{P}(t)$ .
- 2) A distance function  $d(\mathbf{P}, \mathbf{Q})$  between  $\mathbf{P}(t)$  and  $\mathbf{Q}(t)$  in the  $L_2$  norm is minimized.

In general, the integral of a rational Bézier curve over interval  $[0, 1]$  is either very complex or is no analytical expression because the degree of  $n$  grows to be large; thus, analytical solutions of Bézier curve  $\mathbf{Q}(t)$  cannot be directly obtained by the least-squares method. On the other hand, although a rational Bézier curve cannot be expressed by a polynomial precisely, it can be approximately represented as a polynomial Bézier curve of sufficiently high

degree  $m$ . Thus, we can convert the polynomial approximation of a rational Bézier curve to an approximation between two polynomial curves. That is, according to equation (1) and (5), we have

$$\mathbf{P}(t) = \frac{\mathbf{x}(t)}{\omega(t)} \approx \mathbf{Q}(t) \Leftrightarrow \mathbf{x}(t) \approx \omega(t)\mathbf{Q}(t).$$

Let

$$\mathbf{y}(t) = \omega(t)\mathbf{Q}(t),$$

and condition 2) can be modified to the following expression

2a) A distance function  $d(\mathbf{x}, \mathbf{y})$  between  $\mathbf{x}(t)$  and  $\mathbf{y}(t)$  in the  $L_2$  norm is minimized, that is

$$\begin{aligned} \min d(\mathbf{x}, \mathbf{y}) &= \int_0^1 \|\mathbf{x}(t) - \mathbf{y}(t)\|^2 dt \\ &= \int_0^1 \|\mathbf{x}(t) - \omega(t)\mathbf{Q}(t)\|^2 dt \end{aligned} \tag{6}$$

### 3. Algorithm for constrained Bézier approximation

The approximation algorithm is completed through a two-step process. First, we calculate the constrained control points of the approximation Bézier curve  $\mathbf{Q}(t)$  expressed in (6). Next, we calculate the others in terms of the least-squares method.

#### 3.1. Constrained conditions of the approximation curve

According to equation (1), we have

$$\mathbf{x}(t) = \mathbf{P}(t)\omega(t).$$

Taking the derivative of the above equation  $r(= 0, 1, 2)$  times, we obtain

$$\mathbf{x}^{(r)}(t) = \sum_{j=0}^r \binom{r}{j} \omega^{(j)}(t) \mathbf{P}^{(r-j)}(t).$$

Thus, we have the zero, first and second derivatives of  $\mathbf{P}(t)$  at two endpoints ( $t = 0, 1$ ), satisfying the following, respectively

$$\begin{aligned} \mathbf{P}(0) &= \mathbf{p}_0, \quad \mathbf{P}(1) = \mathbf{p}_n, \\ \mathbf{P}'(0) &= n\omega_1(\mathbf{p}_1 - \mathbf{p}_0), \quad \mathbf{P}'(1) = n\omega_{n-1}(\mathbf{p}_n - \mathbf{p}_{n-1}), \\ \mathbf{P}''(0) &= n\{\omega_2(n-1)\mathbf{p}_2 + 2\omega_1(1 - n\omega_1)\mathbf{p}_1 + [2\omega_1(n\omega_1 - 1) + \omega_2(1 - n)]\mathbf{p}_0\}, \\ \mathbf{P}''(1) &= n\{[2\omega_{n-1}(n\omega_{n-1} - 1) + \omega_{n-2}(1 - n)]\mathbf{p}_n + 2\omega_{n-1}(1 - n\omega_{n-1})\mathbf{p}_{n-1} \\ &\quad + (n-1)\omega_{n-1}\mathbf{p}_{n-2}\}. \end{aligned}$$

Then, matching the function value and derivatives up to the second order at both endpoints of  $\mathbf{P}(t)$  and  $\mathbf{Q}(t)$ , the constrained control points of  $\mathbf{Q}(t)$  are

$$\begin{aligned} \mathbf{q}_0 &= \mathbf{p}_0, \quad \mathbf{q}_m = \mathbf{p}_n, \\ \text{if } m \geq 2, \quad \mathbf{q}_1 &= \frac{n}{m}\omega_1\mathbf{p}_1 + \left(1 - \frac{n}{m}\omega_1\right)\mathbf{p}_0, \\ \text{if } m \geq 3, \quad \mathbf{q}_{m-1} &= \left(1 - \frac{n}{m}\omega_{n-1}\right)\mathbf{p}_n + \frac{n}{m}\omega_{n-1}\mathbf{p}_{n-1}, \\ \text{if } m \geq 4, \quad \mathbf{q}_2 &= \frac{1}{m(m-1)}\{n(n-1)\omega_2\mathbf{p}_2 + 2n\omega_1(m - n\omega_1)\mathbf{p}_1 \\ &\quad + [m(m-1) + 2n\omega_1(n\omega_1 - m) + n\omega_2(1 - n)]\mathbf{p}_0\}, \\ \text{if } m \geq 5, \quad \mathbf{q}_{m-2} &= \frac{1}{m(m-1)}\{[2n\omega_{n-1}(n\omega_{n-1} - m) + n\omega_{n-2}(1 - n) + m(m-1)]\mathbf{p}_n \\ &\quad + 2n\omega_{n-1}(m - n\omega_{n-1})\mathbf{p}_{n-1} + n(n-1)\omega_{n-2}\mathbf{p}_{n-2}\}. \end{aligned}$$

Accordingly,  $\mathbf{Q}(t)$  can be rewritten as

$$\mathbf{Q}(t) = \sum_{i=0}^k B_i^m(t) \mathbf{q}_i + \sum_{i=k+1}^{m-h-1} B_i^m(t) \mathbf{q}_i + \sum_{i=m-h}^m B_i^m(t) \mathbf{q}_i, \quad (7)$$

where  $\mathbf{q}_i (i = k+1, k+2, \dots, m-h-1)$  are the unknown control points and

$$k = \begin{cases} 0 & m = 1, \\ 1 & m = 2, 3, \\ 2 & m \geq 4, \end{cases} \quad h = \begin{cases} 0 & m = 1, 2, \\ 1 & m = 3, 4, \\ 2 & m \geq 5. \end{cases}$$

### 3.2. Unconstrained control points of the approximation curve

To minimize the objective function  $d$  expressed by equation (6), the derivatives of  $d$  about the points  $\mathbf{q}_i (i = k+1, \dots, m-h-1)$  should be zero; that is,  $\frac{\partial d}{\partial \mathbf{q}_i} = 0$ . Thus, we have

$$\int_0^1 \mathbf{Q}(t) \omega^2(t) B_l^m(t) dt = \int_0^1 \mathbf{x}(t) \omega(t) B_l^m(t) dt. \quad (8)$$

Substituting (7) into (8), it yields

$$\begin{aligned} & \sum_{i=0}^k \int_0^1 \omega^2(t) B_i^m(t) B_l^m(t) \mathbf{q}_i dt + \sum_{i=k+1}^{m-h-1} \int_0^1 \omega^2(t) B_i^m(t) B_l^m(t) \mathbf{q}_i dt \\ & + \sum_{i=m-h}^m \int_0^1 \omega^2(t) B_i^m(t) B_l^m(t) \mathbf{q}_i dt = \int_0^1 \mathbf{x}(t) \omega(t) B_l^m(t) dt, \end{aligned} \quad (9)$$

By deriving equation (9) based on **properties** 1 and 2, we obtain

$$\begin{aligned}
& \sum_{i=0}^{2n+k} \left( \sum_{j=\max(0, i-2n)}^{\min(k, i)} \frac{\binom{m}{j} \binom{2n}{i-j}}{\binom{2m+2n}{i+l}} W_{i-j} \mathbf{q}_i \right) + \sum_{i=k+1}^{2n+m-h-1} \left( \sum_{j=\max(k+1, i-2n)}^{\min(m-h-1, i)} \frac{\binom{m}{j} \binom{2n}{i-j}}{\binom{2m+2n}{i+l}} W_{i-j} \mathbf{q}_i \right) \\
& + \sum_{i=m-h}^{2n+m} \left( \sum_{j=\max(m-h, i-2n)}^{\min(m, i)} \frac{\binom{m}{j} \binom{2n}{i-j}}{\binom{2m+2n}{i+l}} W_{i-j} \mathbf{q}_i \right) \\
& = \frac{2m+2n+1}{m+2n+1} \sum_{i=0}^{2n} \left( \sum_{j=\max(0, i-n)}^{\min(n, i)} \frac{\binom{n}{j} \binom{n}{i-j}}{\binom{2n+m}{i+l}} \omega_{i-j} \omega_j \mathbf{p}_j \right), \\
& (l = k+1, k+2, \dots, m-h-1), \tag{10}
\end{aligned}$$

where

$$W_i = \sum_{j=\max(0, i-n)}^{\min(n, i)} \frac{\binom{n}{j} \binom{n}{i-j}}{\binom{2n}{i}} \omega_{i-j} \omega_j.$$

To easily evaluate the unknown points  $\mathbf{q}_i (i = k+1, \dots, m-h-1)$ , we rearrange equation (10) to obtain

$$\begin{aligned}
& \sum_{i=k+1}^{m-h-1} \sum_{j=0}^{2n} \frac{\binom{2n}{j} \binom{m}{i}}{\binom{2n+2m}{i+j+l}} W_j \mathbf{q}_i = \frac{2m+2n+1}{2m+n+1} \sum_{i=0}^n \sum_{j=0}^n \frac{\binom{n}{i} \binom{n}{j}}{\binom{2n+m}{i+j+l}} \omega_i \omega_j \mathbf{p}_i \\
& - \sum_{i=0}^k \sum_{j=0}^{2n} \frac{\binom{2n}{j} \binom{m}{i}}{\binom{2n+2m}{i+j+l}} W_j \mathbf{q}_i - \sum_{i=m-h}^m \sum_{j=0}^{2n} \frac{\binom{2n}{j} \binom{m}{i}}{\binom{2n+2m}{i+j+l}} W_j \mathbf{q}_i, \tag{11}
\end{aligned}$$

$$(l = k+1, \dots, m-h-1).$$

Finally, by **property 3**, we conclude that the solutions of linear system (11) are unique. They can be obtained by any methods introduced in [10].



#### 4. Error estimation and implementation

For the convenience of estimation, we use maximum distance  $d_{max}$  to evaluate the approximation results.  $d_{max}$  are the sampling-based various distances between the rational Bézier curve  $\mathbf{P}(t)$  and its approximation curve  $\mathbf{Q}(t)$ . That is,

$$d_{max} = \max_{0 \leq t \leq 1} \|\mathbf{P}(t) - \mathbf{Q}(t)\|$$

**Example 1.** Consider a conic rational Bézier curve  $\mathbf{P}(t)$  with control points  $\mathbf{p}_0 = (0, 0)$ ,  $\mathbf{p}_1 = (0.3, 1.5)$ ,  $\mathbf{p}_2 = (1, 0)$ , and weights  $\omega_0 = 1, \omega_1 = 0.8, \omega_2 = 1$ . A 4th-degree Bézier curve  $\mathbf{Q}(t)$  approximates  $\mathbf{P}(t)$  with the contact order  $(1, 0)$  of continuity at two endpoints. The control points of  $\mathbf{Q}(t)$  are  $(0, 0)$ ,  $(0.12, 0.6)$ ,  $(0.3683, 0.9771)$ ,  $(0.7289, 0.5986)$ ,  $(1, 0)$ . The maximum error distance is  $d_{max} = 2.1 \times 10^{-3}$  (see Figure 1).

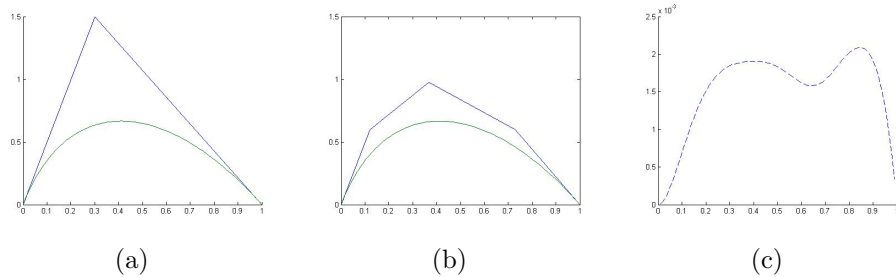


Figure 1: (a) A conic  $\mathbf{P}(t)$  with control points; (b) A 4th-degree Bézier approximation curve  $\mathbf{Q}(t)$  with contact order  $(1, 0)$  of continuity at two endpoints and its control points; (c) Error distance curve

**Example 2.** Consider a cubic section  $\mathbf{P}(t)$  with control points  $\mathbf{p}_0 = (0, 0)$ ,  $\mathbf{p}_1 = (0.2, 1.5)$ ,  $\mathbf{p}_2 = (0.8, 1.5)$ ,  $\mathbf{p}_3 = (1, 0)$ , and weights  $\omega_0 = 1, \omega_1 = 1.2, \omega_2 = 1.5, \omega_3 = 1$ . A 6th-degree Bézier curve  $\mathbf{Q}(t)$  approximates  $\mathbf{P}(t)$  with the contact order  $(2, 2)$  of continuity at two endpoints. The control

points of  $\mathbf{Q}(t)$  are

$(0, 0), (0.12, 0.9), (0.3552, 1.314), (0.5429, 1.49), (0.718, 1.035), (0.85, 1.125), (1.0, 0)$ .

The maximum error distance is  $d_{max} = 1.99 \times 10^{-2}$  (see Figure 2).

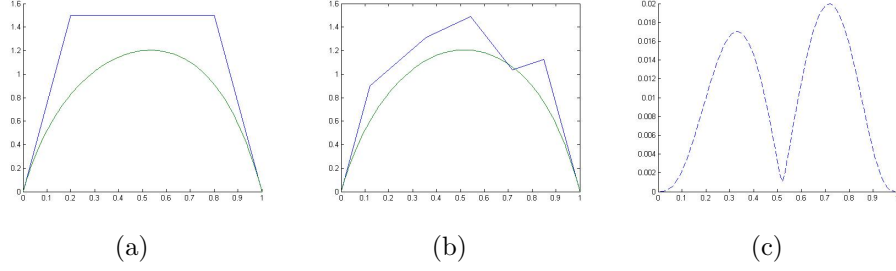


Figure 2: (a) A cubic  $\mathbf{P}(t)$  with control points; (b) A 6th-degree Bézier approximation curve  $\mathbf{Q}(t)$  with contact order  $(2, 2)$  of continuity at two endpoints and its control points; (c) Error distance curve

**Example 3.** Consider a 4th-degree rational Bézier curve  $\mathbf{P}(t)$  with control points  $\mathbf{p}_0 = (0, 0), \mathbf{p}_1 = (0.2, 1.5), \mathbf{p}_2 = (0.4, 1.7), \mathbf{p}_3 = (0.8, 1.5), \mathbf{p}_4 = (1, 0)$  and weights  $\omega_0 = 1, \omega_1 = 1.2, \omega_2 = 1.4, \omega_3 = 1.2, \omega_4 = 1$ . A 6th-degree Bézier curve  $\mathbf{Q}(t)$  approximates  $\mathbf{P}(t)$  with the contact order  $(2, 2)$  of continuity at two endpoints. The control points of  $\mathbf{Q}(t)$  are  $(0, 0), (0.16, 1.2), (0.3008, 1.528), (0.4508, 1.5738), (0.5872, 1.528), (0.84, 1.12), (1, 0)$ . The maximum error distance is  $d_{max} = 1.7 \times 10^{-3}$  (see Figure 3).

**Example 4.** Consider a 4th-degree rational Bézier curve  $\mathbf{P}(t)$  with control points  $\mathbf{p}_0 = (0, 0), \mathbf{p}_1 = (0.2, 1.5), \mathbf{p}_2 = (0.5, 1.0), \mathbf{p}_3 = (0.8, 1.5), \mathbf{p}_4 = (1, 0)$  and weights  $\omega_0 = 1, \omega_1 = 1.2, \omega_2 = 1.4, \omega_3 = 1.2, \omega_4 = 1$ . A 6th-degree Bézier curve  $\mathbf{Q}(t)$  approximates  $\mathbf{P}(t)$  with the contact order  $(1, 1)$  of continuity at two endpoints. The control points of  $\mathbf{Q}(t)$  are  $(0, 0), (0.16, 1.2), (0.3526, 1.1270), (0.5001, 1.2372), (0.6475, 1.127), (0.84, 1.2), (1, 0)$ . The maximum error distance is  $d_{max} = 3.4942 \times 10^{-4}$  (see Figure 4).

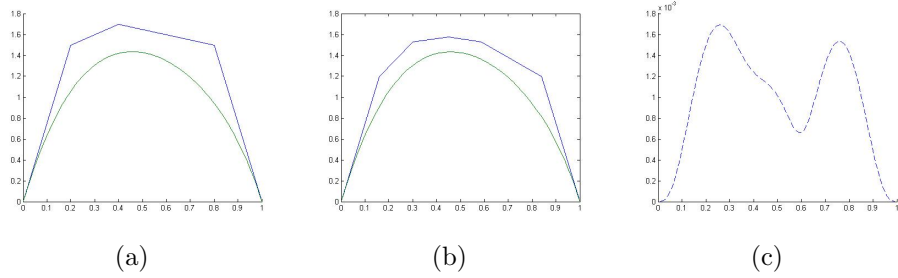


Figure 3: (a) A 4th-degree rational Bézier curve  $P(t)$  with control points; (b) A 6th-degree Bézier approximation curve  $Q(t)$  with contact order  $(2, 2)$  of continuity at two endpoints and its control points; (c) Error distance curve

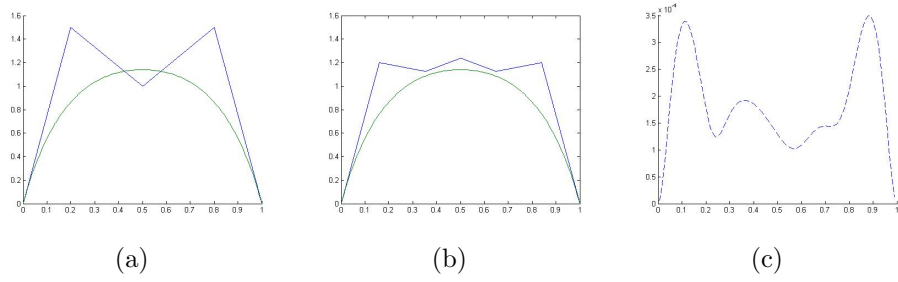


Figure 4: (a) A 4th-degree rational Bézier curve  $P(t)$  with control points; (b) A 6th-degree Bézier approximation curve  $Q(t)$  with contact order  $(1, 1)$  of continuity at two endpoints and its control points; (c) Error distance curve.

## 5. Conclusion and future work

In this paper, we have proposed an algorithm for approximating rational Bézier curves. Compared with other methods, our method is simple, linear and stable. In the future, we will extend this method to solve NURBS approximation problems.

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